國立臺灣海洋大學河海工程學系 2014 工程數學(一) 第一次大考參考解答

系級：________　　學號：________　　姓名：________

1. 試解 $\dfrac{dy(x)}{dx}=\dfrac{px+qy}{rx+sy}$ 在什麼條件或判別式下 (p, q, r, s 之關係)，此 ODE 會正合(exact)? (5%) 此時 $\phi(x,y)=?$ (5%)

2. 給一微分方程式 $x^2y'=y^2+2xy$
   (1) 此微分方程式為線性或非線性? (5%) 正合(exact)或非正合? (5%)
   (2) 試以分離變數法求解。 (Hint: 須先變數變換) (8%)
   (3) 試以正合法解之。(若正合，直接求解$\phi(x,y)$;若非正合，先求出積分因子，再求解$\phi(x,y)$) (8%)
   (4) 試以 Bernoulli 法解之。 (8%)
   (5) 試以全微分法解之。 (8%)

3. 給一 Clairauts 方程式 $y=xy'+y'^2$，試求此微分方程的通解(general solution)與奇解(singular solution)。 (10%)

4. 試解:
   (1) $y'+\dfrac{1}{x}y=\cos x$ (7%)
   (2) $x\dfrac{dy}{dx}+y=x^2y^2$ (7%)
   (3) $y'=\dfrac{xy^2-1}{1-x^2y}$, $y(0)=1$ (7%)
   (4) $3x^2ye^ydx+x^3e^y(y+1)dy=0$ (7%)

5. $y(t)$ 為微分方程 $(e^{\cos y}-t\cdot\sin y)\dfrac{dy}{dt}=1$ 的解，且滿足初始條件 $y(0)=0$，試求 $y(t)=?$ ($y(t)$ 的解可以是隱函數型式) (10%)

1. $\dfrac{dy(x)}{dx}=\dfrac{px+qy}{rx+sy} \quad \Rightarrow (px+qy)dx-(rx+sy)dy=0$

令 $M=(px+qy) \qquad \Rightarrow \dfrac{\partial M}{\partial y}=q$

$N=-(rx+sy)dy \quad \Rightarrow \dfrac{\partial N}{\partial x}=-r$

當 ODE 為正合方程時，則須滿足 $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ 即 $q=-r$

因為是正合型 ODE，可得

$\dfrac{\partial \phi(x,y)}{\partial x}=px+qy \qquad \Rightarrow \phi(x,y)=\dfrac{1}{2}px^2+qxy+f(y)$

$\dfrac{\partial \phi(x,y)}{\partial y}=-(rx+sy) \quad \Rightarrow \phi(x,y)=-rxy-\dfrac{1}{2}sy^2+g(x)$

所以解為 $\phi(x,y)=qxy+\dfrac{1}{2}px^2-\dfrac{1}{2}sy^2=c$

2. (1) 非線性; 非正合

(2) 分離變數法

$x^2y'=y^2+2xy \quad \Rightarrow y'=\dfrac{y^2+2xy}{x^2} \quad \rightarrow$ 為齊次型 ODE

(上下同除 x^2) $\quad \Rightarrow y'=(\dfrac{y}{x})^2+2\dfrac{y}{x}$

令 $u=\dfrac{y}{x} \quad \Rightarrow y=ux \quad \Rightarrow y'=u'x+u$ 代回上式

可得 $u'x+u=u^2+2u \quad \Rightarrow \dfrac{1}{u^2+u}du=\dfrac{1}{x}dx$

$\Rightarrow \int(\dfrac{1}{u}-\dfrac{1}{u+1})du=\int\dfrac{1}{x}dx$

$\Rightarrow \ln|u|-\ln|u+1|=\ln|x|+c_1$

$\Rightarrow \dfrac{u}{x(u+1)}=e^{c_1}$

$\Rightarrow \dfrac{y}{x(x+y)}=e^{c_1} \qquad \Rightarrow \dfrac{x^2}{y}+x=c$

(3) 正合(exact)法

$x^2y'=y^2+2xy \quad \Rightarrow (y^2+2xy)dx-x^2dy=0$

令 $M=y^2+2xy \qquad \Rightarrow \dfrac{\partial M}{\partial y}=2y+2x$

$N=-x^2 \quad \Rightarrow \dfrac{\partial N}{\partial x}=-2x$

由判斷式可知: $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\rightarrow$ 非正合

設積分因子 $\mu = \mu(y)$ $\Rightarrow \frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) = -\frac{2}{y}$

$\therefore \int \frac{1}{\mu} d\mu = \int \frac{1}{M}(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) dy = -\int \frac{2}{y} dy$

$\Rightarrow \ln|\mu| = -2\ln|y|$ $\Rightarrow \mu = \frac{1}{y^2}$

將微分方程同乘積分因子，可得 $(1 + 2\frac{x}{y})dx - \frac{x^2}{y^2}dy = 0$ 為正合

$\therefore M = \frac{\partial \phi(x,y)}{\partial x} = 1 + 2\frac{x}{y}$ $\Rightarrow \phi(x,y) = x + \frac{x^2}{y} + f(y)$

$N = \frac{\partial \phi(x,y)}{\partial y} = -\frac{x^2}{y^2}$ $\Rightarrow \phi(x,y) = \frac{x^2}{y} + g(x)$

故微分方程解為 $\phi(x,y) = x + \frac{x^2}{y} = c$

(4) Bernoulli 法

$x^2 y' = y^2 + 2xy$ $\Rightarrow y' - \frac{2}{x}y = \frac{1}{x^2}y^2$ $\rightarrow$ 為 Bernoulli 型微分方程

$$\Rightarrow y^{-2}y' - \frac{2}{x}y^{-1} = \frac{1}{x^2}$$

令 $u = y^{-1}$ $\Rightarrow u' = -y^{-2}y'$ 代回上式

可得 $u' + \frac{2}{x}u = -\frac{1}{x^2}$ $\rightarrow$ 為一階線性微分方程

積分因子為 $\mu = e^{\int \frac{2}{x}dx} = x^2$

同乘積分因子後可得 $x^2u' + 2xu = -1$ $\Rightarrow d(x^2u) = -dx$

$\Rightarrow x^2u = -x + c$

$\Rightarrow \frac{x^2}{y} + x = c$

(5) 全微分法

$x^2 y' = y^2 + 2xy$ $\Rightarrow (y^2 + 2xy)dx - x^2dy = 0$

$\Rightarrow y^2dx + x(2ydx - xdy) = 0$

$\Rightarrow y^2dx + xd(y^{-1}x^2)\cdot\frac{y^2}{x} = 0$

$\Rightarrow dx + d(y^{-1}x^2) = 0$

(兩邊同時積分) $\Rightarrow x + \frac{x^2}{y} = c$

3. $y = xy' + y'^2$

令 $p = y' \Rightarrow y = xp + p^2$

將兩邊對 x 微分可得 $y' = p + xp' + 2pp'$

$$\Rightarrow p = p + xp' + 2pp'$$

$$\Rightarrow (x + 2p)p' = 0$$

由 $p' = 0 \Rightarrow y' = p = c$ 代回 $y = xy' + y'^2$

可得 $y = xc + c^2$（通解）

由 $x + 2p = 0 \Rightarrow y' = p = -\dfrac{x}{2}$ 代回 $y = xy' + y'^2$

可得 $y = -\dfrac{x^2}{4}$（奇解）

4. (1) $y' + \dfrac{1}{x} y = \cos x \Rightarrow xy' + y = x\cos x$

$$\Rightarrow d(xy) = x\cos x\, dx$$

$$\Rightarrow xy = x\sin x + \cos x + c$$

$$\Rightarrow y = \sin x + \frac{1}{x}\cos x + \frac{c}{x}$$

(2) $x\dfrac{dy}{dx} + y = x^2y^2 \Rightarrow xdy + ydx = x^2y^2dx$

$$\Rightarrow d(xy) = x^2y^2dx$$

$$\Rightarrow \frac{1}{x^2y^2}d(xy) = dx$$

$$\Rightarrow -\frac{1}{xy} = x + c \quad \Rightarrow y = -\frac{1}{x}\cdot\frac{1}{x+c}$$

(3) $y' = \dfrac{xy^2 - 1}{1 - x^2y} \Rightarrow (xy^2 - 1)dx - (1 - x^2y)dy = 0$

$$\Rightarrow xy(ydx + xdy) - dx - dy = 0$$

$$\Rightarrow xy\, d(xy) - dx - dy = 0$$

$$\Rightarrow \frac{1}{2}x^2y^2 - x - y = c$$

又 $y(0) = 1 \Rightarrow c = -1$

故此微分方程解為 $x + y - \dfrac{1}{2}x^2y^2 = 1$

(4) $3x^2ye^y dx + x^3e^y(y+1)dy = 0$

$\Rightarrow \dfrac{3}{x}dx + \dfrac{y+1}{y}dy = 0 \quad \Rightarrow \displaystyle\int \frac{3}{x}dx + \int \frac{y+1}{y}dy = c_1$

$$\Rightarrow 3\ln x + y + \ln y = c_1$$

$$\Rightarrow 3\ln x + \ln e^y + \ln y = c_1 \quad \Rightarrow x^3ye^y = c$$

5. $(e^{\cos y}-t\cdot\sin y)\dfrac{dy}{dt}=1 \quad \Rightarrow (e^{\cos y}-t\cdot\sin y)\dfrac{dy}{dt}=1$

$$\Rightarrow \frac{dy}{dt}=\frac{1}{e^{\cos y}-t\cdot\sin y}$$

$$\Rightarrow \frac{dt}{dy}=e^{\cos y}-t\cdot\sin y$$

$$\Rightarrow \frac{dt}{dy}+\sin y\cdot t=e^{\cos y}$$

當 y 為 t 函數時，則此題不易求解。

反之，將 t 視為 y 函數時，即 $t(y)$ ，則此題可視為 $t(y)$ 的一階線性 ODE

積分因子為 $\mu=e^{\int \sin y\,dy}=e^{-\cos y}$

同乘積分因子可得 ODE 為 $e^{-\cos y}\dfrac{dt}{dy}+e^{-\cos y}\cdot\sin y\cdot t=e^{-\cos y}\cdot e^{\cos y}$

$$\Rightarrow \frac{d}{dy}(e^{-\cos y}\cdot t)=1$$

$$\Rightarrow e^{-\cos y}\cdot t=y+c$$

代入初始條件 $y(0)=0$，即當 $t=0$ 時 $y=0$ 。

可得 $c=0$ ，故解為 $e^{-\cos y}\cdot t-y=0$

此題亦可以正合法求解:

$(e^{\cos y}-t\cdot\sin y)\dfrac{dy}{dt}=1 \quad \Rightarrow dt+(t\cdot\sin y-e^{\cos y})dy=0$

$M=1 \qquad \Rightarrow \dfrac{\partial M}{\partial y}=0$

$N=t\cdot\sin y-e^{\cos y} \qquad \Rightarrow \dfrac{\partial N}{\partial t}=\sin y$

$\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial t}\Rightarrow$ not exact

$\mu=\mu(y) \quad \Rightarrow \dfrac{1}{\mu}d\mu=\dfrac{1}{M}(\dfrac{\partial N}{\partial t}-\dfrac{\partial M}{\partial y})dy \quad \Rightarrow \int\dfrac{1}{\mu}d\mu=\int\dfrac{1}{M}(\dfrac{\partial N}{\partial t}-\dfrac{\partial M}{\partial y})dy$

$$\Rightarrow \ln\mu=\int\sin y\,dy=-\cos y$$

$$\Rightarrow \mu=e^{-\cos y}$$

同乘積分因子可得 $e^{-\cos y}dt+e^{-\cos y}(t\cdot\sin y-e^{\cos y})dy=0$

$M=\dfrac{\partial\phi(t,y)}{\partial t}=e^{-\cos y} \quad \Rightarrow \phi(t,y)=t\cdot e^{-\cos y}+f(y)$

$N=\dfrac{\partial\phi(t,y)}{\partial y}=e^{-\cos y}(t\cdot\sin y-e^{\cos y})\Rightarrow \phi(t,y)=\int e^{-\cos y}(t\cdot\sin y-e^{\cos y})dy$

$$=\int(t\cdot\sin y\cdot e^{-\cos y}-1)dy$$

$$= -\int (t \cdot e^{-\cos y}) d \cos y - y$$

$$= t \cdot e^{-\cos y} - y + g(t)$$

$\therefore \phi(t, y) = t \cdot e^{-\cos y} - y = c$

代入初始條件 $y(0) = 0$，即當 $t = 0$ 時 $y = 0$ 。

可得 $c = 0$ ，故解為 $e^{-\cos y} \cdot t - y = 0$